Regular Cayley maps on dihedral groups with the smallest kernel

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Abstract

Let $\mathcal{M} = \mathrm{CM}(D_n, X, p)$ be a regular Cayley map on the dihedral group D_n of order $2n, n \geq 2$, and let π be the power function associated with $\mathcal{M}$. In this paper it is shown that the kernel $\mathrm{Ker}(\pi)$ of the power function π is a dihedral subgroup of D_n and if $n \neq 3$, then the kernel $\mathrm{Ker}(\pi)$ is of order at least 4. Moreover, all $\mathcal{M}$ are classified for which $\mathrm{Ker}(\pi)$ is of order 4. In particular, besides 4 sporadic maps on $4, 4, 8$ and 12 vertices respectively, two infinite families of non-t-balanced Cayley maps on D_n are obtained.

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1 Introduction

In this paper all groups are finite, and all graphs are finite, simple and connected. For a graph Γ, we let $V(\Gamma)$, $E(\Gamma)$, $D(\Gamma)$, and $\mathrm{Aut}(\Gamma)$ denote the vertex set, the edge set, the dart (or arc) set, and the full group of automorphisms of Γ, respectively. By a *map* with an underlying graph Γ we mean a triple $\mathcal{M} = (\Gamma; R, T)$, where R is a permutation of the dart set $D(\Gamma)$ whose orbits coincide with the sets of darts initiating in the same vertex, and T is an involution of $D(\Gamma)$ whose orbits coincide with sets of darts with the same underlying edge. The permutations R and T are called the *rotation* and the *dart-reversing involution* of $\mathcal{M}$, respectively. Given two maps $\mathcal{M}_i = (\Gamma_i; R_i, T_i)$, $i = 1, 2$, an *isomorphism* $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a bijection $\Phi : D(\Gamma_1) \to D(\Gamma_2)$ such that $\Phi R_1 = R_2 \Phi$ and $\Phi T_1 = T_2 \Phi$. In particular, if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, then Φ is called an *automorphism*, and the group of all automorphisms of $\mathcal{M}$ will be denoted by $\mathrm{Aut}(\mathcal{M})$. It is easily seen

that $\mathrm{Aut}(\mathcal{M})$ acts semi-regularly on the dart set $D(\Gamma)$, and in the case when this action is also transitive the map $\mathcal{M}$ is called *regular*. In what follows the map $(\Gamma; R, T)$ will be written as the pair $(\Gamma; R)$ because T is uniquely defined by Γ. For more information on regular maps we refer the reader to the survey paper [10].

Let G be a group and let X be a generating set of G such that $X = X^{-1}$ and $1_G \notin X$, where 1_G denotes the identity of G. The *Cayley graph* $\mathrm{Cay}(G, X)$ is the graph with vertex set G and with edges in the form $\{g, gx\}, g \in G, x \in X$. The *left multiplication* L_g induced by $g \in G$ is the permutation of G defined by $L_g(h) = gh$ for any $h \in G$. We set $L(G) = \{L_g : g \in G\}$. It is clear that $L(G) \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$. Let p be a cyclic permutation of X. The *Cayley map* $\mathrm{CM}(G, X, p)$ is the map $(\Gamma; R)$ with underlying graph $\Gamma = \mathrm{Cay}(G, X)$ and rotation R defined by $R : (g, gx) \mapsto (g, gp(x)), g \in G, x \in X$. It can be easily checked that for every $g \in G$, $L_g R = R L_g$ and $L_g T = T L_g$, so $L(G)$ is also a subgroup of $\mathrm{Aut}(\mathcal{M})$ acting regularly on the vertex set. Two Cayley maps $\mathcal{M}_i = \mathrm{CM}(G_i, X_i, p_i), i = 1, 2$, are called *equivalent*, denoted by $\mathcal{M}_1 \equiv \mathcal{M}_2$, if there exists a group isomorphism $\phi : G_1 \to G_2$ mapping X_1 to X_2 such that $\phi p_1 = p_2 \phi$. Equivalent Cayley maps are isomorphic as maps. The converse, however, does not hold in general, i.e., there exist isomorphic Cayley maps which are not equivalent.

The class of cyclic groups is the only class of finite groups on which all regular Cayley maps have been classified due to the work of Conder and Tucker [4]. Regarding other groups, only partial classifications are known (see, e.g. [7, 8, 9, 11, 15, 16, 17]). For more information on regular Cayley maps, the reader is referred to [1, 5, 12, 13, 14].

In this paper we focus on regular Cayley maps on dihedral groups. The dihedral group of order $2n$ for $n \geq 2$ will be denoted by D_n. A complete classification of regular Cayley maps on dihedral groups have been given in [15] for balanced maps; in [8] for t-balanced maps with $t > 1$; in [7] for non-balanced maps with n being an odd number; and in [17] for maps of skew-type 3. Recall that a Cayley map $\mathcal{M} = \mathrm{CM}(G, X, p)$ is *t-balanced* if $p(x)^{-1} = p^t(x^{-1})$ for every $x \in X$. In particular, if $t = 1$ then $\mathcal{M}$ is called *balanced*, and if $t = -1$, then $\mathcal{M}$ is called *anti-balanced*. Let π be the power function associated with a regular Cayley map $\mathcal{M} = \mathrm{CM}(G, X, p)$ (for the definition of π, see 2.1). Let $\mathrm{Ker}(\pi) = \{g \in G : \pi(g) = 1\}$ be the kernel of the power function π. Following [16], we also say that $\mathcal{M}$ is of *skew-type k when* $|G : \mathrm{Ker}(\pi)| = k$. If $\mathcal{M}$ is t-balanced, then it was proved to be of skew-type at most 2 in [3, 13]. More precisely, $t = 1$ holds if and only if $\mathrm{Ker}(\pi) = G$ (see [13]); and if $t > 1$ and $G = D_n$, then $\mathrm{Ker}(\pi)$ is a dihedral subgroup of D_n of index 2. In this context, the papers [8, 15, 17] deal with regular Cayley maps $\mathcal{M} = \mathrm{CM}(D_n, X, p)$ having a large kernel.

In this paper we consider the other extreme case, i.e., the associated kernel is as small as possible. We are going to prove that, if $\mathcal{M} = \mathrm{CM}(D_n, X, p)$ is a regular Cayley map with associated power function π, then either $\mathcal{M}$ is the embedding of the octahedron into the sphere and $|\mathrm{Ker}(\pi)| = 2$, or $|\mathrm{Ker}(\pi)| \geq 4$ (see Theorem 4.3). Moreover, we are also going to determine those maps $\mathcal{M}$ for which $|\mathrm{Ker}(\pi)| = 4$. In this paper we set $D_n = \langle a, b \mid a^n = b^2 = baba = 1 \rangle$ and $C_n = \langle a \rangle$. Note that if $n > 2$, then C_n is the unique cyclic subgroup of D_n of order n.

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Theorem 1.1. *Let $\mathcal{M}$ be a regular Cayley map on D_n such that $|\operatorname{Ker}(\pi)| = 4$ for the associated power function π. Then exactly one of the following holds:*

(1) $n = 2$, and $\mathcal{M} \equiv \operatorname{CM}(D_2, \{a, b\}, (a, b))$ or $\operatorname{CM}(D_2, \{a, b, ab\}, (a, b, ab))$.

(2) $n = 4$, and $\mathcal{M} \equiv \operatorname{CM}(D_4, \{a, a^{-1}, b\}, (a, b, a^{-1}))$.

(3) $n = 6$, and $\mathcal{M} \equiv \operatorname{CM}(D_6, \{a, a^{-1}, ab, a^{-1}b\}, (a, a^{-1}, ab, a^{-1}b)$.

(4) $n = 2m$, $n \geq 6$, $\mathcal{M} \equiv \operatorname{CM}(D_n, a\langle a^2 \rangle \cup b\langle a^2 \rangle, p)$ with

$$p = (b, a, a^2b, a^3, a^4b, \ldots, a^{n-2}b, a^{n-1}).$$

(5) $n = 2m$, $8 \mid n$, $\mathcal{M} \equiv \operatorname{CM}(D_n, a\langle a^2 \rangle \cup b\langle a^2 \rangle, p)$ with

$$p = (b, a, a^{m+2}b, a^3, a^4b, \ldots, a^{m-2}b, a^{n-1}).$$

2 Preliminaries

In this section we collect all concepts and results needed in this paper.

2.1 Skew-morphisms of finite groups. For a finite group G, let $\psi : G \to G$ be a permutation of the underlying set G of order r (in the full symmetric group $\operatorname{Sym}(G)$) and let $\pi : G \to \{1, \ldots, r\}$ be any function. The permutation ψ is a *skew-morphism* of G with *power function* π if $\psi(1_G) = 1_G$, and $\psi(gh) = \psi(g)\psi^{\pi(g)}(h)$ for all $g, h \in G$. Skew-morphisms were defined by Jajcay and Širáň in [5], where the following theorem was shown:

Theorem 2.1. *A Cayley map $\mathcal{M} = \operatorname{CM}(G, X, p)$ is regular if and only if there exists a skew-morphism ψ of G such that $\psi(x) = p(x)$ for all $x \in X$.*

The skew-morphism ψ and its power function π in the above theorem are uniquely determined by the regular Cayley map $\mathcal{M}$. In what follows these will be referred to as the *skew-morphism (power function) associated with $\mathcal{M}$*. More precisely, for a given regular Cayley map $\operatorname{CM}(G, X, p)$, the associated skew-morphism ψ is of order $|X|$, and the distribution of the values of π on X is given by the following formula (see [5]):

$$\pi(x) \equiv \chi(\psi(x)) - \chi(x) + 1 \pmod{|X|} \text{ for any } x \in X, \tag{1}$$

where $\chi(x)$ is the smallest non-negative integer such that $p^{\chi(x)}(x) = x^{-1}$ (notice that $x^{-1} \in X$ as $X = X^{-1}$ holds).

The *kernel* of the power function π is defined by $\operatorname{Ker}(\pi) = \{g \in G : \pi(g) = 1\}$. The following lemma shows some basic properties (see [5]):

Lemma 2.2. *Let ψ be a skew morphism of G and let π be the corresponding power function of ψ.*

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(1) $\text{Ker}(\pi)$ *is a subgroup of* G.

(2) $\pi(g) = \pi(h)$ *if and only if* g *and* h *belong to the same right coset of* $\text{Ker}(\pi)$.

(3) $\pi(gh) \equiv \sum_{i=0}^{\pi(g)-1} \pi(\psi^i(h))(mod\ r)$, *where* r *is the order of* ψ.

For a Cayley map $\mathcal{M} = \text{CM}(G, X, p)$, we will denote by $\text{Aut}(\mathcal{M})_{1_G}$ the stabilizer of the vertex 1_G in the group $\text{Aut}(\mathcal{M})$ in its action on the vertices. Notice that if $\mathcal{M}$ is regular, then $\text{Aut}(\mathcal{M})_{1_G}$ is generated by the skew-morphism ψ associated with $\mathcal{M}$, and so $\text{Aut}(\mathcal{M})$ admits the factorization $\text{Aut}(\mathcal{M}) = L(G)\,\text{Aut}(\mathcal{M})_{1_G} = L(G)\,\langle\psi\rangle$.

2.2 G-arc-regular Cayley graphs on dihedral groups. Let Γ be a graph and let $G \leq \text{Aut}(\Gamma)$. Then Γ is called *G-arc-regular* if G is regular on the dart set $D(\Gamma)$. Clearly, if $\mathcal{M} = (\Gamma; R)$ is a regular map, then the underlying graph Γ is $\text{Aut}(\mathcal{M})$-arc-regular.

Let $n = 2m$, m is an odd number. For the rest of the paper we set $\mathbb{D}_n$ for the group $\mathbb{D}_n = (D_m \times D_m) \rtimes \langle\sigma\rangle$, where σ is an involution of $\mathbb{D}_n$ which acts on $D_m \times D_m$ by switching the coordinates, i.e., $\mathbb{D}_n = \langle D_m \times D_m, \sigma\rangle$, and

$$\sigma(d_1, d_2)\sigma = (d_2, d_1) \text{ for all } (d_1, d_2) \in D_m \times D_m. \tag{2}$$

The *core* of a subgroup $A \leq B$ in a group B, denoted by $\text{Core}_B(A)$, is the largest normal subgroup of B contained in A. The subgroup A is *core-free* in B if $\text{Core}_B(A)$ is trivial.

The following result of Kovács et al. [7] will be one of our main tools in this paper (see [7, Theorem 2.8]):

Theorem 2.3. *Let* $\Gamma = \text{Cay}(\mathbb{D}_n, S)$ *be a connected G-arc-regular graph such that* $L(\mathbb{D}_n) \leq G$, *and every cyclic subgroup of* $L(\mathbb{D}_n)$ *of order* n *is core-free in* G. *Then one of the following holds:*

(1) $n = 1$, $\Gamma \cong K_2$, *and* $G \cong S_2$,

(2) $n = 2$, $\Gamma \cong K_4$, *and* $G \cong A_4$,

(3) $n = 3$, $\Gamma \cong K_{2,2,2}$, *and* $G \cong S_4$,

(4) $n = 4$, $\Gamma \cong Q_3$, *and* $G \cong S_4$,

(5) $n = 2m$, m *is an odd number,* $\Gamma \cong K_{n,n}$, *and* $G \cong \mathbb{D}_n$. *Moreover, the subgroup of* $\mathbb{D}_n$ *corresponding to* $L(\mathbb{D}_n)$ *is contained in* $D_m \times D_m$.

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2.3 Quotient Cayley maps. Let $\mathcal{M} = \mathrm{CM}(G, X, p)$ be a regular Cayley map. Suppose, in addition, that there exists a subgroup $N \leq G$ such that N is normal in G and the the set of N-cosets is a block system of $\mathrm{Aut}(\mathcal{M})$. In what follows it will be simply said that G/N is a block system for $\mathrm{Aut}(\mathcal{M})$. Furthermore, we set $X/N = \{Nx : x \in X\}$. Clearly, X/N is a generating subset of the factor group G/N and $X/N = (X/N)^{-1}$. Also, since $\mathrm{Cay}(G, X)$ is $\mathrm{Aut}(\mathcal{M})$-arc-regular, no element of X belongs to N, and so $1_{G/N} \notin X/N$.

There is an action of $\mathrm{Aut}(\mathcal{M})$ on the set of blocks, i.e., on G/N. For $g \in \mathrm{Aut}(\mathcal{M})$, we let $\bar{g}$ denote the action of g on G/N, and for a subgroup $H \leq \mathrm{Aut}(\mathcal{M})$ set $H^{G/N} = \{\bar{g} : g \in H\}$. Notice that $(L_g)^{G/N} = L_{Ng}$ for every $g \in G$. Let us write $X = \{x_1, \ldots, x_k\}$ and $p = (x_1, x_2, \ldots, x_k)$. Then it follows that the cycle $p^{G/N} := (x_1 N, x_2 N, \ldots, x_k N)$ is well-defined (see [7]); and so is the Cayley map $\mathrm{CM}(G/N, X/N, p^{G/N})$. The latter map is called the *quotient* of $\mathcal{M}$ with respect to the block system G/N, and it will be also denoted by $\mathcal{M}/N$. We note that the quotient map $\mathcal{M}/N$ coincides with the so called *Cayley-quotient* induced by the normal subgroup N which was defined by Zhang [17], and in the same paper $\mathcal{M}$ is also referred to as the *Cayley-cover* of $\mathcal{M}/N$. We collect below some properties (see [7, Corollary 3.5]):

Lemma 2.4. *Let $\mathcal{M} = \mathrm{CM}(G, X, p)$ be a regular Cayley map with associated skew-morphism ψ and power function π, and let $N \leq G$ be a normal subgroup in G and G/N is a block system for $\mathrm{Aut}(\mathcal{M})$. Then the following hold:*

(1) $\mathcal{M}/N = \mathrm{CM}(G/N, X/N, p^{G/N})$ is also regular.

(2) $\mathrm{Aut}(\mathcal{M}/N) = \mathrm{Aut}(\mathcal{M})^{G/N}$.

(3) The skew-morphisms associated with $\mathcal{M}/N$ is equal to $\psi^{G/N}$.

(4) The order $|\langle\psi\rangle| \leq |N| \cdot |\langle\psi^{G/N}\rangle|$, and equality holds if and only if X is a union of N-cosets.

(5) The power function $\pi^{G/N}$ associated with $\mathcal{M}/N$ satisfies

$$\pi^{G/N}(Ng) \equiv \pi(g)\left(mod \left|\langle\psi^{G/N}\rangle\right|\right) \text{ for every } g \in G.$$

3   Regular Cayley maps with a given group

Let G be a finite group, H be a non-trivial subgroup of G, and let x, y be elements in G such that $y \neq 1_G$. We say that the ordered quadruple (G, H, x, y) is *admissible* if the following properties hold:

- $G = HY$ and $|H \cap Y| = 1$, where $Y = \langle y \rangle$;

- Y is core-free in G;

- $G = \langle Y, x \rangle$ and $YxY = Yx^{-1}Y$.

Every admissible quadruple (G, H, x, y) gives rise to a regular Cayley map on H defined as follows.

First, recall that the *coset graph* $\Gamma = \mathrm{Cos}(G, Y, YxY)$ has vertex set G/Y, the set of left Y-cosets in G, and its edges are in the form $\{g_1Y, g_2Y\}$, $g_1, g_2 \in G$ and $g_1^{-1}g_2 \in YxY$. Note that the edges are well-defined because of the condition $YxY = Yx^{-1}Y$. Also, the condition $G = \langle Y, x \rangle$ makes Γ to be connected. Since the group Y is core-free in G, the action of G on the set G/Y is a faithful permutation representation of G. Furthermore, the dart set $D(\Gamma)$ is, in fact, equal to the orbit of the dart (Y, xY) under G.

Now, using that $G = HY$ and $|H \cap Y| = 1$, there is a bijection from G/Y onto H. Observe that this bijection induces an isomorphism from Γ to the Cayley graph $\mathrm{Cay}(H, X)$, where X is the subset of H defined by

$$X = \{h \in H : hY \subseteq YxY\}.$$

Notice that X is the unique subset of H that satisfies $XY = YxY$.

Also, by the above bijection we obtain a faithful permutation representation of G on H. More precisely, an element $g \in G$ acts on H by letting $g(h)$ to be the unique element of H for which

$$g(h)Y = ghY, \ \text{where } g \in G, h \in H. \tag{3}$$

Furthermore, X becomes the orbit of x under Y in the above action. Hence we can define the cyclic permutation p of X as

$$p = \big(x, y(x), y^2(x), \ldots, y^{|Y|-1}(x)\big).$$

Now, the Cayley map $\mathrm{CM}(H, X, p)$ will be called the *Cayley map induced by* (G, H, x, y), and in what follows we will write $\mathrm{CM}(G, H, x, y)$ for $\mathrm{CM}(H, X, p)$.

Lemma 3.1. *Let (G, H, x, y) be an admissible quadruple and let $\mathcal{M} = \mathrm{CM}(G, H, x, y)$ be the Cayley map on H induced by (G, H, x, y). Then the following hold:*

(1) $\mathcal{M}$ is regular, and $\mathrm{Aut}(\mathcal{M}) \cong G$.

(2) If $\alpha : G \to \widehat{G}$ is an isomorphism, then the quadruple $(\alpha(G), \alpha(H), \alpha(x), \alpha(y))$ is also admissible; moreover, $\mathcal{M}$ and $\mathcal{M}(\alpha(G), \alpha(H), \alpha(x), \alpha(y))$ are equivalent.

Proof. Let us consider the action of G on H defined in (3). Let ψ_y denote the permutation of H describing the action of y. Then $p(x) = \psi_y(x)$ for every $x \in X$, and thus case (1) of the lemma follows if we can prove that ψ_y is a skew-morphism of H (see Theorem 2.1 and the remark after Lemma 2.2).

Notice that the permutation of H describing the action of $h \in H$ is equal to the left multiplication L_h. Thus the permutation subgroup corresponding to the action of G factorizes as $L(H)\langle \psi_y \rangle$, in particular, $L(H)\langle \psi_y \rangle = \langle \psi_y \rangle L(H)$.

We compute next $\psi_y(1_H)$. By (3), $\psi_y(1_H)Y = y1_HY = Y$, hence $\psi_y(1_H) \in H \cap Y$, and so $\psi_y(1_H) = 1_H$.

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Pick an arbitrary $h \in H$. Since $\langle \psi_y \rangle L(H) = L(H) \langle \psi_y \rangle$, $\psi_y L_h = L_{h'} \psi_y^i$ for a unique $h' \in H$ and a unique $i \in \{1, \ldots, |Y|\}$. Notice that i depends entirely on h, and thus we may define the function $\pi : H \to \{1, \ldots, |Y|\}$ by letting $\pi(h) = i$. Also, $(\psi_y L_h)(1_H) = \psi_y(h)$ and $(L_{h'} \psi_y^i)(1_H) = h'$. These give that $h' = \psi_y(h)$. Thus if $h_1, h_2 \in H$, then we may write

$$\psi_y(h_1 h_2) = (\psi_y L_{h_1})(h_2) = (L_{\psi_y(h_1)} \psi_y^{\pi(h_1)})(h_2) = \psi_y(h_1) \psi_y^{\pi(h_1)}(h_2),$$

showing that ψ_y is indeed a skew-morphism of H with power function π. So case (1) of the lemma is proved.

We turn to case (2). It is obvious that all defining axioms of an admissible quadruple are preserved by α, and so $(\alpha(G), \alpha(H), \alpha(x), \alpha(y))$ is also admissible.

By definition, $\mathrm{CM}(G, H, x, y) = \mathrm{CM}(H, X, p)$, where X is the unique subset of H satisfying $XY = YxY$; and for $z \in X$, $p(z)$ is the unique element in H satisfying $p(z)Y = yzY$. Also, $\mathrm{CM}(\alpha(G), \alpha(H), \alpha(x), \alpha(y)) = \mathrm{CM}(\alpha(H), \widehat{X}, \widehat{p})$, where $\widehat{X}$ is the unique subset of $\alpha(H)$ satisfying $\widehat{X}\alpha(Y) = \alpha(Y)\alpha(x)\alpha(Y)$; and for $\widehat{z} \in \widehat{X}$, $\widehat{p}(\widehat{z})$ is the unique element in $\alpha(H)$ satisfying $\widehat{p}(\widehat{z})\alpha(Y) = \alpha(y)\widehat{z}\alpha(Y)$.

Obviously, $\alpha(X)\alpha(Y) = \alpha(Y)\alpha(x)\alpha(Y)$. This implies that $\widehat{X} = \alpha(X)$, i.e., α maps X onto $\widehat{X}$. For any $z \in X$, it follows from $p(z)Y = yzY$ that

$$\alpha(p(z))\alpha(Y) = \alpha(p(z)Y) = \alpha(yzY) = \alpha(y)\alpha(z)\alpha(Y).$$

Since $\alpha(p(z)) \in \alpha(H)$, we have $\alpha(p(z)) = \widehat{p}(\alpha(z))$. We conclude that $\alpha p = \widehat{p}\alpha$, and so $\mathrm{CM}(H, X, p)$ and $\mathrm{CM}(\alpha(H), \widehat{X}, \widehat{p})$ are equivalent. Case (2) of the lemma is proved. $\square$

Remark 3.2. Let φ be an arbitrary permutation of H such that $\varphi(1_H) = 1_H$. Now, it becomes apparent from the above proof that a sufficient condition for φ to be a skew-morphism of H is that $\varphi L(H) \subseteq L(H)\langle \varphi \rangle$.

Lemma 3.3. *Let $\mathcal{M} = \mathrm{CM}(H, X, p)$ be a regular Cayley map such that $\mathrm{Aut}(\mathcal{M}) \cong G$. Now there exists an admissible quadruple (G, K, x, y) such that $H \cong K$ and $\mathcal{M}$ and $\mathrm{CM}(G, K, x, y)$ are equivalent.*

Proof. Let ψ be the skew-morphism associated with $\mathcal{M}$ and let $\Psi = \langle \psi \rangle$. The group $\mathrm{Aut}(\mathcal{M})$ factorizes as $\mathrm{Aut}(\mathcal{M}) = L(H)\Psi$ such that $|L(H) \cap \Psi| = 1$, see the remark after Lemma 2.2. Also, by Ψ being the stabilizer of the vertex 1_H in $\mathrm{Aut}(\mathcal{M})$, the group Ψ is core-free in $\mathrm{Aut}(\mathcal{M})$. Let us fix an element $x_1 \in X$. Using that X is a generating set of H, it is not hard to show that $\langle \Psi, L_{x_1} \rangle$ is transitive on H. Using this and that $\langle \Psi, L_{x_1} \rangle \geq \Psi = \mathrm{Aut}(\mathcal{M})_{1_H}$, we obtain that $\langle \Psi, L_{x_1} \rangle = \mathrm{Aut}(\mathcal{M})$. Since $X = X^{-1}$, x_1^{-1} also belongs to X, and hence $\Psi L_{x_1} \Psi = \Psi L_{x_1}^{-1} \Psi$. We may summarize all these as the quadruple $(\mathrm{Aut}(\mathcal{M}), L(H), L_{x_1}, \psi)$ is admissible.

Denote by L the isomorphism from H to $L(H)$ defined by $L : h \mapsto L_h$. Let h be an element in H such that $L_h \Psi \subset \Psi L_{x_1} \Psi$. Then $L_h = \psi^i L_{x_1} \psi^j$ for some integers i and j, and so $h = L_h(1_H) = (\psi^i L_{x_1} \psi^j)(1_H) = \psi^j(x_1)$. As X is the orbit of x_1 under Ψ we see that $h \in X$. On the other hand, for any $h \in X$, $h = \psi^k(x_1)$ for some integer

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k. This implies that the product $L_h^{-1}\psi^k L_{x_1}$ fixes 1_H, and so $L_h^{-1}\psi^k L_{x_1}\Psi = \Psi$, namely $L_h\Psi \subset \Psi L_{x_1}\Psi$. Thus

$$\mathrm{CM}(\mathrm{Aut}(\mathcal{M}), L(H), L_{x_1}, \psi) = \mathrm{CM}(L(H), L(X), \widehat{p}), \tag{4}$$

where for $z \in X$, $\widehat{p}(L_z)$ is the unique element of $L(H)$ satisfying $\widehat{p}(L_z)\Psi = \psi L_z\Psi$. As $(\psi L_z)(1_H) = \psi(z)$ and $L_{p(z)}(1_H) = p(z) = \psi(z)$, we have $L_{p(z)}\Psi = \psi L_z\Psi$. Since $L_{p(z)} \in L(H)$, $L_{p(z)} = \widehat{p}(L_z)$. We conclude that $Lp = \widehat{p}L$, and so $\mathrm{CM}(H, X, p)$ and $\mathrm{CM}((L(H), L(X), \widehat{p})$ are equivalent.

Let α be an isomorphism $\alpha : \mathrm{Aut}(\mathcal{M}) \to G$. Then by (4) and Lemma 3.1(2), $\mathrm{CM}(L(H), L(X), \widehat{p})$ is also equivalent with $\mathrm{CM}(G, \alpha(L(H)), \alpha(L_{x_1}), \alpha(\psi))$. The lemma follows by choosing $K = \alpha(L(H))$, $x = \alpha(L_{x_1})$, and $y = \alpha(\psi)$. $\qquad\square$

We combine the above lemmas with Theorem 2.3 to have the following theorem.

Theorem 3.4. *Let $\mathcal{M}$ be a regular Cayley map on D_n such that the unique cyclic subgroup of $L(D_n)$ of order n is core-free in $\mathrm{Aut}(\mathcal{M})$ with $n \geq 5$. Then $n = 2m$, m is an odd number, $\mathcal{M} \equiv \mathrm{CM}(D_n, a\langle a^2\rangle \cup b\langle a^2\rangle, p)$, where*

$$p = (b, a, a^2 b, a^3, a^4 b, \ldots, a^{n-2} b, a^{n-1}).$$

Proof. The Cayley graph $\mathrm{Cay}(D_n, X)$ is $\mathrm{Aut}(\mathcal{M})$-arc-regular, and thus Theorem 2.3 is applicable. This gives that $n = 2m$, m is an odd number, $\mathrm{Aut}(\mathcal{M}) \cong \mathbb{D}_n$, and the subgroup of $\mathbb{D}_n$ corresponding to $L(D_n)$ is contained in $D_m \times D_m$. The proof of Lemma 3.3 yields that

$$\mathcal{M} \equiv \mathrm{CM}(\mathbb{D}_n, H, x_1, y_1) \tag{5}$$

for some admissible quadruple $(\mathbb{D}_n, H, x_1, y_1)$ such that $H \cong D_n$ and $H \leq D_m \times D_m$.

Let us fix c as a generator of the cyclic subgroup of D_m of order m, and an involution $r \in D_m$ such that $r \notin \langle c\rangle$. Define the subgroup $D \leq D_m \times D_m$ as

$$D = \{(d, r^i) : d \in D_m, i \in \{0, 1\}\}.$$

Then $D \cong D_m \times \mathbb{Z}_2 \cong D_n$. It was proved in [6, Proposition 3.2] that, every two subgroups of $D_m \times D_m$, which are isomorphic to D_n, are conjugate in $\mathbb{D}_n$. Thus $D = H^g$ for some $g \in \mathbb{D}_n$. Apply Lemma 3.1(2) to $\mathrm{CM}(\mathbb{D}_n, H, x_1, y_1)$ with letting α be the inner automorphism $\alpha : z \mapsto z^g, z \in \mathbb{D}_n$. This and (5) imply that

$$\mathcal{M} \equiv \mathrm{CM}(\mathbb{D}_n, D, x_2, y_2), \ x_2 = x_1^g \text{ and } y_2 = y_1^g. \tag{6}$$

Let $Y_2 = \langle y_2\rangle$. Recall that Y_2 is core-free in $\mathbb{D}_n$, $\mathbb{D}_n = DY_2$, and $|D \cap Y_2| = 1$. Since $\mathbb{D}_n = DY_2$ and $D \leq (D_m \times D_m)$, $Y_2 \nleq (D_m \times D_m)$. We obtain that $|Y_2| = |\mathbb{D}_n : D| = n$, and $|Y_2 \cap (D_m \times D_m)| = |Y_2|/2 = m$. Since m is odd, the unique involution y_2^m in Y_2 is in the form $y_2^m = (d, d')\sigma$ for some $(d, d') \in D_m \times D_m$. Then $((d, d')\sigma)^2 = y_2^{2m} = 1_G$, hence $d' = d^{-1}$. This gives $y_2^m = (d, 1)\sigma(d, 1)^{-1}$, i.e., y_2^m is conjugate to σ by $(d, 1) \in D$. This and (6) imply that

$$\mathcal{M} \equiv \mathcal{M}(\mathbb{D}_n, D, x, y), \ x = x_2^{(d,1)} \text{ and } y = (y_2)^{(d,1)}. \tag{7}$$

Notice that $y^m = \sigma$. Also, $y = (d_1, d_2)\sigma$ for some $(d_1, d_2) \in D_m \times D_m$. As y commutes with σ and has order n, we find that $d_1 = d_2 = c^\ell$ for some integer ℓ with $\gcd(\ell, m) = 1$. Thus we have

$$Y = \langle y \rangle = \{(c^i, c^i)\sigma^j : i \in \{0, 1, \ldots, m-1\}, j \in \{0, 1\}\}. \tag{8}$$

We compute next the skew-morphism ψ_y of D obtained from the action of y defined in (3). We set $a = (c, r)$ and $b = (r, 1)$, and so $D = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$. Let $i \in \{0, 1, \ldots, m-1\}$. Using (2) and (8), we may write

$$
\begin{aligned}
ya^{2i}Y &= (c^\ell, c^\ell)\sigma(c^{2i}, 1)Y = (c^{-2i}, 1)(c, c)^{2i+\ell}\sigma Y = (c^{n-2i}, 1)Y = a^{n-2i}Y, \\
y(a^{2i+1}b)Y &= (c^\ell, c^\ell)\sigma(c^{2i+1}r, r)Y = (c^\ell r, c^{\ell+2i+1}r)\sigma Y = (c^{-2i-1}r, r)Y = a^{n-2i-1}bY, \\
ya^{2i+1}Y &= (c^\ell, c^\ell)\sigma(c^{2i+1}, r)Y = (c^\ell r, c^{\ell+2i+1})\sigma Y = (c^{2\ell+2i+1}r, 1)Y = a^{2i+1+2\ell+m}bY, \\
y(a^{2i}b)Y &= (c^\ell, c^\ell)\sigma(c^{2i}r, 1)Y = (c^\ell, c^{\ell+2i}r)\sigma Y = (c^{2\ell+2i}, r)Y = a^{2i+2\ell+m}Y.
\end{aligned}
$$

Therefore, ψ_y is given by

$$\psi_y(a^j) = \begin{cases} a^{n-j} & \text{if } j \text{ is even} \\ a^{j+2\ell+m}b & \text{if } j \text{ is odd} \end{cases} \qquad \psi_y(a^j b) = \begin{cases} a^{j+2\ell+m} & \text{if } j \text{ is even} \\ a^{n-j}b & \text{if } j \text{ is odd.} \end{cases}$$

Notice that the set $X = a\langle a^2 \rangle \cup b\langle a^2 \rangle$ is the only orbit of ψ_y which generates D_n. This and (7) imply that $\mathcal{M} \equiv \mathrm{CM}(D, X, p_1)$, where $p_1(z) = \psi_y(z)$ for every $z \in X$.

Since $\gcd(\ell, m) = 1$ and m is odd, it follows that, $\gcd(2\ell + m, m) = 1$. This implies that $\gcd(2\ell + m, n) = 1$ because $2\ell + m$ is odd. Thus there is an integer ℓ' such that $(2\ell + m)\ell' \equiv 1 \pmod{n}$. Define the automorphism α of D by letting $\alpha(a) = a^{\ell'}$ and $\alpha(b) = b$. Now, it is easily seen that α induces an equivalence from $\mathrm{CM}(D, X, p_1)$ to $\mathrm{CM}(D, X, p)$, where $p = (b, a, a^2 b, a^3, a^4 b, \ldots, a^{n-2}b, a^{n-1})$. The theorem is proved. $\square$

4   The smallest kernel of regular Cayley maps on D_n

In this section, we show that every kernel of regular Cayley maps on D_n is a dihedral subgroup of D_n. Furthermore such kernel is of order at least 4 except only one regular Cayley maps on D_3. We start with a simple observation.

Lemma 4.1. *Let* $\mathcal{M} = \mathrm{CM}(G, X, p)$ *be a regular Cayley map with associated skew-morphism* ψ *and power function* π. *Then* $\mathrm{Ker}(\pi) \cong L(G) \cap L(G)^\psi$.

Proof. Let L be the isomorphism $L : G \to L(G)$ defined by $L : g \mapsto L_g$. Since ψ is a skew-morphism with power function π, $\psi L_g = L_{\psi(g)}\psi^{\pi(g)}$ holds in $\mathrm{Aut}(\mathcal{M})$ for every $g \in G$. The equivalences follow:

$$g \in \mathrm{Ker}(\pi) \iff L_g^{\psi^{-1}} = \psi L_g \psi^{-1} \in L(G) \iff L_g \in L(G)^\psi.$$

Therefore, L maps the group $\mathrm{Ker}(\pi)$ onto $L(G) \cap L(G)^\psi$. The lemma is proved. $\square$

Corollary 4.2. *Let $\mathcal{M} = \mathrm{CM}(D_n, X, p)$ be a regular Cayley map with associated skew-morphism ψ and power function π and let $M = C_n \cap \mathrm{Ker}(\pi)$. If $|M| > 2$, then $L(M)$ is normal in $\mathrm{Aut}(\mathcal{M})$.*

Proof. Let L be the isomorphism $L : G \to L(G)$ defined by $L : g \mapsto L_g$. We have proved above that L maps M to $L(D_n) \cap L(D_n)^\psi$. Thus $L(M) \leq L(D_n)^\psi$, and hence $L(M)^{\psi^{-1}} \leq L(D_n)$. Since $|M| > 2$, $L(M)$ is the unique cyclic subgroup of $L(D_n)$ of order $|M|$, and we conclude that $L(M)^\psi = L(M)$. Therefore, $L(M)$ is normal in $\langle L(D_n), \psi \rangle = \mathrm{Aut}(\mathcal{M})$. $\qquad\square$

Theorem 4.3. *Let $n \geq 2$, and $\mathcal{M} = \mathrm{CM}(D_n, X, p)$ be a regular Cayley map with associated skew-morphism ψ and power function π. Now*

(1) $\mathrm{Ker}(\pi)$ is a dihedral subgroup of D_n.

(2) Either $\mathcal{M}$ is the embedding of the octahedron into the sphere and $|\mathrm{Ker}(\pi)| = 2$, or $|\mathrm{Ker}(\pi)| \geq 4$.

Proof. Let N be the subgroup of C_n such that $L(N)$ is the core of $L(C_n)$ in $\mathrm{Aut}(\mathcal{M})$. Notice that $N \leq \mathrm{Ker}(\pi)$. If N is trivial, then the results hold by Theorem 2.3. So assume that N is non-trivial, namely $|N| \geq 2$. Since $N \leq \mathrm{Ker}(\pi)$, it suffices to show that $\mathrm{Ker}(\pi)$ is a dihedral subgroup of D_n, namely $\mathrm{Ker}(\pi) \cap (D_n \setminus C_n) \neq \emptyset$.

Let us consider the largest subgroup $H \leq C_n$ containing N such that D_n/H is a block system for $\mathrm{Aut}(\mathcal{M})$. Now if $H = C_n$, then $L(D_n)$ is normal in $\mathrm{Aut}(\mathcal{M})$, $\mathrm{Ker}(\pi) = D_n$, which prove the results. Hence assume that $H < C_n$. We set $\Gamma = \mathrm{Cay}(D_n, X)$ and $\Gamma/H = \mathrm{Cay}(D_n/H, X/H)$.

Recall that Γ/H is the underlying graph of the quotient map $\mathcal{M}/H$ (see 2.3 and Lemma 2.4). Now $\mathcal{M}/H$ is a regular Cayley map on the dihedral group D_n/H, and Γ/H is $\mathrm{Aut}(\mathcal{M}/H)$-arc-regular. Since $\mathrm{Aut}(\mathcal{M}/H) = \mathrm{Aut}(\mathcal{M})^{D_n/H}$, it follows by the maximality of H that the core of $L(C_n/H)$ is trivial in $\mathrm{Aut}(\mathcal{M}/H)$. Theorem 2.3 is applicable, in particular, Γ/H and $\mathrm{Aut}(\mathcal{M}/H)$ are described in one of cases (2)-(5) of Theorem 2.3.

If Γ/H and $\mathrm{Aut}(\mathcal{M}/H)$ correspond to (2) or (3) in Theorem 2.3, then there exists a $x \in X \cap (D_n \setminus C_n)$ such that $\psi(x) \in D_n \setminus C_n$. This means that $x \in \mathrm{Ker}(\pi)$, and hence $\mathrm{Ker}(\pi)$ is a dihedral subgroup of D_n.

Suppose that Γ/H and $\mathrm{Aut}(\mathcal{M}/H)$ correspond to (4) or (5) in Theorem 2.3. Then there exists $x \in X \cap C_n$ such that $\psi(x), \psi^{-1}(x^{-1}) \in X \cap (D_n \setminus C_n)$. If $p^i(x^{-1}) = x$, then $\pi(x) = \pi(\psi^{-1}(x^{-1})) = i+1$. Since $x \in C_n$ and $\psi^{-1}(x^{-1}) \in D_n \setminus C_n$, $\mathrm{Ker}(\pi)$ is a dihedral subgroup of D_n by Lemma 2.2(2). This completes the proof of the theorem. $\qquad\square$

We study next the Cayley maps defined in cases (4) and (5) of Theorem 1.1, respectively.

Lemma 4.4. *Let $n = 2m$, $n \geq 6$, $\mathcal{M} = \mathrm{CM}(D_n, a\langle a^2 \rangle \cup b\langle a^2 \rangle, p)$, where $p = (b, a, a^2 b, a^3, a^4 b, \ldots, a^{n-2} b, a^{n-1})$. Then the following hold:*

(1) $\mathcal{M}$ is regular, and its associated skew-morphism ψ is given by

$$\psi(a^j) = \begin{cases} a^{-j} & \text{if } j \text{ is even} \\ a^{j+1}b & \text{if } j \text{ is odd} \end{cases} \qquad \psi(a^j b) = \begin{cases} a^{j+1} & \text{if } j \text{ is even} \\ a^{-j}b & \text{if } j \text{ is odd.} \end{cases}$$

(2) Let N be the subgroup of C_n such that $L(N)$ is the core of $L(C_n)$ in $\mathrm{Aut}(\mathcal{M})$. Then $|N| \leq 2$, and $|N| = 1$ if and only if m is odd.

(3) The associated power function π has kernel $\mathrm{Ker}(\pi) \cong \mathbb{Z}_2^2$.

(4) Let $\Pi = \{\pi(g) : g \in D_n\}$. Now $\Pi = \{2i + 1 : i \in \{0, 1, \ldots, m-1\}\}$. If $4 \mid n$, then

$$\pi(x) \equiv -1 (mod\ 4) \iff x \in a\langle a^2 \rangle \cup b\langle a^2 \rangle. \tag{9}$$

Proof. We consider all points step by step.

(1): By Remark 3.2, it is sufficient to show that $\psi L(D_n) \subseteq L(D_n)\langle \psi \rangle$.

It can be checked by definition that ψ^2 is skew-morphism of D_n whose power function takes 1 on all $x \in \langle a^2, b \rangle$, and -1 on the remaining elements. Hence ψ^2 acts on $\langle a^2, ab \rangle$ as the identical permutation, and on the rest as the left multiplication L_{a^2}. Direct computations yield

$$L_{a^2 b} \psi L_a \psi^{-1} = \psi^2 \text{ and } L_{a^{-1}} \psi L_b \psi^{-1} = \psi^{-2}.$$

Now we may write $\psi L_{a^j} \psi^{-1} = (\psi L_a \psi^{-1})^j = (L_{a^2 b} \psi^2)^j \in L(D_n)\langle \psi^2 \rangle$, and thus $\psi L_{a^j} \in L(D_n)\langle \psi^2 \rangle \psi \subseteq L(D_n)\langle \psi \rangle$. Similarly, $\psi L_{a^j b} \psi^{-1} = (\psi L_{a^j} \psi^{-1})(\psi L_b \psi^{-1}) = (L_{a^2 b} \psi^2)^j L_a \psi^{-2} \in L(D_n)\langle \psi^2 \rangle$, and so $\psi L_{a^j b} \in L(D_n)\langle \psi^2 \rangle \psi \subseteq L(D_n)\langle \psi \rangle$.

(2): Suppose for the moment that m is even. It is then easily seen that $L_{a^m}\psi = \psi L_{a^m}$, and thus $L_{a^m} \in L(N)$. By this observation it suffices to prove that, if $|N| \geq 2$, then m is even and $|N| = 2$.

Let $L(N)$ be generated by L_{a^k} where k is a divisor of $n = 2m$. Now $(L_{a^k})^\psi \in L(N)$ and $(L_{a^k})^\psi(1_H) = (\psi^{-1} L_{a^k} \psi)(1_H) = \psi^{-1}(a^k)$. This implies in turn that, k is even, $(L_{a^k})^\psi(1_H) = a^{-k}$, and so $(L_{a^k})^\psi = L_{a^{-k}}$. Then $a^{-k}b = L_{a^{-k}}(b) = (L_{a^k})^\psi(b) = a^k b$. Thus $k = m$ is even, and so $|N| = 2$.

(3): Let $M = C_n \cap \mathrm{Ker}(\pi)$. By Lemma 4.1, $L(M)^\psi$ is contained in $L(D_n)$. If $|M| > 2$, then $L(M)^\psi = L(M)$ follows by (1), hence $M \leq N$. This gives $|N| > 2$, which is a contradiction with (2). Therefore, $|M| \leq 2$, and so $|\mathrm{Ker}(\pi)| \leq 4$. We finish the proof of (3) by showing that $\{1, a^m, ba, ba^{m+1}\} \leq \mathrm{Ker}(\pi)$.

For $x \in a\langle a^2 \rangle \cup b\langle a^2 \rangle$, let $\chi(x)$ be the smallest non-negative integer such that $p^{\chi(x)}(x) = x^{-1}$. Then by (1), $x \in \mathrm{Ker}(\pi)$ if and only if $\chi(\psi(x)) = \chi(x)$. This shows that a^m and $a^{m-1}b = ba^{m+1}$ are in $\mathrm{Ker}(\pi)$ if m is odd, and thus so are 1 and ba.

Let m be even. Then $L_{a^m}\psi = \psi L_{a^m}$, hence $a^m \in \mathrm{Ker}(\pi)$. It remain to show that $ba \in \mathrm{Ker}(\pi)$. Now

$$a = \psi(b) = \psi(baa^{-1}) = \psi(ba)\psi^{\pi(ba)}(a^{-1}) = ab\psi^{\pi(ba)}(a^{-1}).$$

From this, $\psi^{\pi(ba)}(a^{-1}) = b$, and so $\pi(ba) = 1$.

(4): Note that $\mathrm{Ker}(\pi) = \{1, a^m, ba, ba^{m+1}\}$. Since for any $x \in X$, $\pi(x) \equiv \chi(\psi(x)) - \chi(x) + 1 \pmod{n}$, one can check $\pi(a^{2i}b) \equiv n - 4i - 1 \pmod{n}$ and $\pi(a^{2i+1}) \equiv 4i + 3 \pmod{n}$ for any $i = 0, \ldots, m-1$. Furthermore for any $j = 0, \ldots, m-1$,

$$\pi(a^{2j+2}) = \pi(a \cdot a^{2j+1}) \equiv \pi(a^{2j+1}) + \pi(\psi(a^{2j+1})) + \pi(\psi^2(a^{2j+1})) \equiv 4j + 5 \pmod{n}.$$

Therefore, $\Pi = \{2i + 1 : i \in \{0, 1, \ldots, m-1\}\}$ and if $4 \mid n$, then

$$\pi(x) \equiv -1 \pmod 4 \iff x \in a\langle a^2 \rangle \cup b\langle a^2 \rangle.$$

$\square$

The next lemma can be derived in the similar way as above, and hence its proof is omitted.

Lemma 4.5. *Let $n = 2m$, $8 \mid n$, $\mathcal{M} = \mathrm{CM}(D_n, a\langle a^2 \rangle \cup b\langle a^2 \rangle, p)$, where $p = (b, a, a^{m+2}b, a^3, a^4 b, \ldots, a^{m-2}b, a^{n-1})$. Then the following hold:*

(1) $\mathcal{M}$ is regular, and its associated skew-morphism ψ is given by

$$\psi(a^j) = \begin{cases} a^{\frac{j}{2}m - j} & \text{if } j \text{ is even} \\ a^{j+1+\frac{j+1}{2}m}b & \text{if } j \text{ is odd} \end{cases} \qquad \psi(a^j b) = \begin{cases} a^{j+1+\frac{j}{2}m} & \text{if } j \text{ is even} \\ a^{\frac{j+1}{2}m - j}b & \text{if } j \text{ is odd.} \end{cases}$$

(2) Let N be the subgroup of C_n such that $L(N)$ is the core of $L(C_n)$ in $\mathrm{Aut}(\mathcal{M})$. Then $|N| = 2$.

(3) The associated power function π has kernel $\mathrm{Ker}(\pi) \cong \mathbb{Z}_2^2$.

(4) Let $\Pi = \{\pi(g) : g \in D_n\}$. Then $\Pi = \{2i + 1 : i \in \{0, 1, \ldots, m-1\}\}$, and

$$\pi(x) \equiv -1 \pmod 4 \iff x \in a\langle a^2 \rangle \cup b\langle a^2 \rangle. \tag{10}$$

5   Proof of Theorem 1.1

In this section we set

- $\mathcal{M} = \mathrm{CM}(D_n, X, p)$ is a regular Cayley map;

- ψ and π are the associated skew-morphism and power-function respectively;

- N is the subgroup of C_n such that $L(N)$ is the core of $L(C_n)$ in $\mathrm{Aut}(\mathcal{M})$;

- $n = 2m$, $m \geq 1$, and $T = \langle a^m \rangle$ is the subgroup of C_n of order 2.

Lemma 5.1. *Let $n \geq 8$, $|\mathrm{Ker}(\pi)| = 4$ and N be non-trivial. Now the following hold:*

(1) $N = T$ and D_n/T is a block system for $\mathrm{Aut}(\mathcal{M})$.

(2) $|\operatorname{Ker}(\pi^{D_n/T})| = 4.$

(3) m *is even.*

Proof. We set $G = \operatorname{Aut}(\mathcal{M})$ and $\Gamma = \operatorname{Cay}(D_n, X)$. The cases (1)-(3) are considered separately.

(1): Since $N \le \operatorname{Ker}(\pi)$, $|N| = 2$ by Theorem 4.3 and thus $T = N$. Since $N \trianglelefteq G$, $T \trianglelefteq G$ also holds, and (1) follows.

(2): Let K be the kernel of the action of $\operatorname{Aut}(\mathcal{M})$ on D_n/T. Let Y_1, Y_2 be two T-cosets such that (Y_1, Y_2) is an arc of Γ/T. Let S be the set of arcs from Y_1 to Y_2 in Γ. It is easily seen that K is regular on S, hence $K \cong \mathbb{Z}_2^i$ for $i \in \{1, 2\}$. Let $H \le C_n$ such that $T \le H$ and $KL(H)/K$ is the core of $KL(C_n)/K$ in G/K. This implies that $KL(H) \trianglelefteq G$. Define the subgroup $M = \langle x^2 : x \in KL(H)\rangle$. Clearly, this is characteristic in $KL(H)$. As $K \cong \mathbb{Z}_2^i, i \in \{1, 2\}$ and $L(T) \le K$, the group $KL(H)$ is $L(H)$ or it can be written as a semidirect product $L(H) \rtimes \mathbb{Z}_2$. This implies that $M = L(H^+)$, where $H^+ < H$ and $|H : H^+| = 2$. Since $L(H^+)$ is characteristic in $KL(H)$ and $KL(H)$ is normal in G, $L(H^+)$ is normal in G. Thus $L(H^+) \le L(T)$, $|H| \le 4$, and so $|\operatorname{Core}_{G/K}(KL(C_n)/K)| \le 2$. Using also that $G^{D_n/T} \cong G/K$ and $KL(C_n)/K \cong L(C_n)^{D_n/T}$, we conclude that the core of $L(C_n)^{D_n/T}$ in $G^{C_n/T}$ has order at most 2. Since $\operatorname{Aut}(\mathcal{M}/T) = G^{D_n/T}$ and $L(C_n/T) = L(C_n)^{D_n/T}$, this together with Corollary 4.2 imply that $|\ker(\pi^{D_n/T})| \le 4$. On the other hand, applying Theorem 4.3 to $\mathcal{M}/T$, we find that $|\ker(\pi^{D_n/T})| \ge 4$ since $n \ge 8$. Hence $|\ker(\pi^{D_n/T})| = 4$.

(3): Let K be the kernel of the action of $\operatorname{Aut}(\mathcal{M})$ on D_n/T. We have shown in the proof of (2) that $|\operatorname{Core}_{G/K}(KL(C_n)/K)| \le 2$. If equality holds, then $m = |KL(C_n)/K|$ is even. If $|\operatorname{Core}_{G/K}(KL(C_n)/K)| = 1$, then Theorem 2.3 gives that m is even because $n \ge 8$. The lemma is proved. $\qquad\square$

Lemma 5.2. *Let $n \ge 8$ and $|\operatorname{Ker}(\pi)| = 4$. Now the following hold:*

(1) $X = a\langle a^2\rangle \cup b\langle a^2\rangle$, *and ψ switches the sets $a\langle a^2\rangle$ and $b\langle a^2\rangle$.*

(2) Let $\Pi = \{\pi(g) : g \in D_n\}$. Then $\Pi = \{2i + 1 : i \in \{0, 1, \ldots, m - 1\}\}$.

(3) For any $g \in D_n$ and for any $a^k \in X$, $\pi(ga^k) = \pi(g) + \pi(a^k) - 1(\bmod\, n)$.

(4) If $4 \mid n$, then $\pi(x) \equiv -1(\bmod\, 4)$ if and only if $x \in a\langle a^2\rangle \cup b\langle a^2\rangle$.

Proof. We prove the lemma by induction on n. If $n \le 16$, all statements can be checked directly, using the catalog of small regular maps in [2]. Therefore assume that $n > 16$, and that the lemma holds for any n' such that $8 \le n' < n$. We consider all cases (1)-(4) step by step.

(1): If N is trivial, then (1) follows from Theorems 2.3 and 3.4.

Let N be non-trivial. Lemma 5.1 together with the induction hypothesis imply that $X/T = (a\langle a^2\rangle \cup b\langle a^2\rangle)/T$, and $\psi^{D_n/T}$ switches the sets $(a\langle a^2\rangle)/T$ and $(b\langle a^2\rangle)/T$. Notice that for every $x \in a\langle a^2\rangle \cup b\langle a^2\rangle$, $|Tx \cap X|$ is the same positive constant, say c, which

does not depend on x. If $c = 1$, then $|X| = n/2$. Let us consider the action of $\mathrm{Aut}(\mathcal{M})$ on the set of right $L(D_n)$-cosets by right multiplication. This has degree $|\langle\psi\rangle|$. For any $d \in D_n$, $L(D_n)\psi L_d = L(D_n)\psi^{\pi(d)}$. This shows that the orbit of the coset $L(D_n)\psi$ under $L(D_n)$ is equal to $\{L(D_n)\psi^{\pi(d)} : d \in D_n\}$, and thus it has size $|\Pi|$. Clearly, $L(D_n)$, as a coset, is fixed by every permutation in $L(D_n)$. All these imply that $|\Pi| < |\langle\psi\rangle|$. Since $|\mathrm{Ker}(\pi)| = 4$, $|\Pi| = n/2$, and we may write

$$\frac{n}{2} = |\Pi| < |\langle\psi\rangle| = |X| = \frac{n}{2}.$$

This is a contradiction, and so $c = 2$, $X = a\langle a^2\rangle \cup b\langle a^2\rangle$. Since $\psi^{D_n/T}$ switches the sets $(a\langle a^2\rangle)/T$ and $(b\langle a^2\rangle)/T$, ψ also switches the sets $a\langle a^2\rangle$ and $b\langle a^2\rangle$.

(2): If N is trivial, then (2) follows from Theorem 3.4 and Lemma 4.4(4).

Let N be non-trivial. Lemma 5.1 together with the induction hypothesis imply that, for any $Tg \in D_n/T$, $\pi^{D_n/T}(Tg)$ is odd. Notice that Lemma 5.1(3) gives that $4 \mid n$, and hence $\psi^{D_n/T}$ is of even order. Now, by Lemma 2.4(5), for any $g \in D_n$, $\pi(g)$ is also odd, which implies that $\Pi = \{2i + 1 : i \in \{0, 1, \ldots, m - 1\}\}$.

(3): Since ψ switches the sets $a\langle a^2\rangle$ and $b\langle a^2\rangle$, for any $x \in b\langle a^2\rangle$, we obtain by the formula in (1) that

$$\pi(x) + \pi(\psi(x)) \equiv 2(\mathrm{mod}\ n).$$

Using this property and Lemma 2.2(3), one can say that for any $g \in D_n$ and $a^k \in X$,

$$
\begin{aligned}
\pi(ga^k) &\equiv \sum_{i=0}^{\pi(g)-1} \pi(\psi^i(a^k)) \\
&\equiv \pi(a^k) + \big(\pi(\psi(a^k)) + \pi(\psi^2(a^k))\big) + \cdots + \big(\pi(\psi^{\pi(g)-2}(a^k)) + \pi(\psi^{\pi(g)-1}(a^k))\big) \\
&\equiv \pi(a^k) + \pi(g) - 1(\mathrm{mod}\ n)
\end{aligned}
$$

because $\pi(g)$ is odd by case (2).

(4): If N is trivial, then (4) follows from Theorem 3.4 and Lemma 4.4(4).

Let N be non-trivial. Assume at first that $\mathcal{M}/T$ is a regular Cayley map on D_n/T such that the core of $L(C_n/T)$ is trivial in $\mathrm{Aut}(\mathcal{M}/T)$. Now $\frac{n}{2} = m = 2m'$ with odd m' by Theorem 3.4. Since for any $a^k b \in D_n \setminus C_n$, there exists a group automorphism ϕ such that $\phi(a^k b) = b$, we can assume that $\psi(a^{-1}) = b$. Now $\psi(b) = a$ or $\psi(b) = a^{m+1}$. Suppose that $\psi(b) = a^{m+1}$. Since $a^m \in N$ and N is normalized by ψ, $a^m \in \mathrm{Ker}(\pi)$ and $\psi(a^m) = a^m$. Then $\psi(a^{m-1}) = a^m b$ and $\psi(a^m b) = a$. These imply that

$$\pi(a^{-1}) = \pi(a^{m-1}) = \pi(b) = \pi(a^m b) = m - 1 \ \ \text{and} \ \ \pi(a) = \pi(a^{m+1}) = m + 3.$$

By (3) we find $\pi(ga) \equiv \pi(g) + m + 2(\mathrm{mod}\ n)$ for every $g \in D_n$. Since $n \equiv 0(\mathrm{mod}\ 4)$ and $m \equiv 2(\mathrm{mod}\ 4)$, $\pi(g) \equiv \pi(ga)(\mathrm{mod}\ 4)$ for every $g \in D_n$. This, however, contradicts (2). Therefore, $\psi(b) = a$, and hence

$$\pi(a^{-1}) = \pi(a^{m-1}) = \pi(b) = \pi(a^m b) = -1 \ \ \text{and} \ \ \pi(a) = \pi(a^{m+1}) = 3.$$

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These imply that $\pi(a^2) = 2\pi(a) - 1 = 5$. For any $x \in a\langle a^2\rangle$, $\pi(a^2 x) = \pi(x) + \pi(a^2) - 1 = \pi(x) + 4$. Hence $\{3, 7, \ldots, n-1\} \subset \{\pi(x) : x \in X\}$. Since $X = a\langle a^2\rangle \cup b\langle a^2\rangle$ is a union of some right cosets of $\mathrm{Ker}(\pi) = \{1, a^m, ba, ba^{m+1}\}$, $|\{\pi(x) : x \in X\}| = \frac{|X|}{4} = \frac{n}{4}$. Hence $\{\pi(x) : x \in X\} = \{3, 7, \ldots, n-1\}$.

For the remaining case, let $\mathcal{M}/T$ be a regular Cayley map on D_n/T such that the core of $L(C_n/T)$ is non-trivial in $\mathrm{Aut}(\mathcal{M}/T)$. Thus $n/2$ is divisible by 4, see Lemma 5.1(3). On the other hand, the order of $\psi^{D_n/T}$ is equal to $|X/T| = n/2$, and hence $4 \mid |\langle\psi^{D_n/T}\rangle|$. By the induction hypothesis, $\{\pi^{D_n/T}(Tx) : Tx \in X/T\} = \{3, 7, \ldots, \frac{n}{2} - 1\}$. Note that for any $Tx \in X/T$, $\pi^{D_n/T}(Tx) \equiv -1 \pmod 4$. By Lemma 2.4(5) and the fact that $4 \mid |\langle\psi^{D_n/T}\rangle|$, for any $x \in X$, $\pi(x) \equiv -1 \pmod 4$. Since X is a union of some right cosets of $\mathrm{Ker}(\pi)$, $\{\pi(x) : x \in X\} = \{3, 7, \ldots, n-1\}$. $\qquad\square$

Proof of Theorem 1.1. Let $\mathcal{M}$ be the regular Cayley map given in the theorem. If $n \le 6$, then one can check, using the catalog of small regular maps [2], that $\mathcal{M}$ is given in one of cases (1)-(3) in Theorem 1.1. Let $n > 6$.

If n is odd, then it follows from [7, Theorem 3.2] that the core of $L(C_n)$ in $\mathrm{Aut}(\mathcal{M})$ is equal to either $L(C_n)$, or the subgroup of $L(C_n)$ of index 3. Since $n > 6$, the core is non-trivial of odd order, implying that $|\mathrm{Ker}(\pi)| \ne 4$, which is a contradiction. Thus $n = 2m$ and $n \ge 8$. We follow the notations set at the beginning of this section. We proceed by induction on n. If $n \le 14$, then the statement can be checked directly, using the catalog of small regular maps [2]. Therefore, assume that $n \ge 16$ and that the lemma holds for any n' such that $8 \le n' < n$.

We set $G = \mathrm{Aut}(\mathcal{M})$, $\Gamma = \mathrm{Cay}(D_n, X)$, and $\Pi = \{\pi(g) : g \in D_n\}$. Because of Theorem 3.4, we may assume that N is non-trivial. By Lemma 5.2(2), $\Pi = \{2i + 1 : i \in \{0, 1, \ldots, m-1\}\}$. Now D_n/T is a block system for G, and $|\mathrm{Ker}(\pi^{D_n/T})| = 4$. Since $n \ge 16$, m is even by Lemma 5.1(3). The induction hypothesis gives that, up to equivalence of $\mathcal{M}$, we may write $p^{D_n/T} = (Tb, Ta, Ta^2 b, \ldots, Ta^{m-1})$ or $p^{D_n/T} = (Tb, Ta, Ta^{2+m/2}b, \ldots, Ta^{m-1})$. By Lemma 5.2(4), there exists $x \in X$ such that $\pi(x) = n - 1$. This implies that $\pi(a^k) = n - 1$ for some $a^k \in X$. Thus $\pi(Ta^k) = m - 1$, see Lemma 2.4(5). If $m = 2m'$ with odd m', then k equals $m-1$ or $n-1$. If m is a multiple of 4, then $k \in \{\frac{1}{2}m - 1, m - 1, \frac{3}{2}m - 1, n - 1\}$. In either cases, there exists a group automorphism ϕ such that $\phi(a^k) = a^{-1}$, $\phi(\psi(a^k)) = b$ and $\phi(\psi^2(a^k)) = a$ since k and n are relative prime and $\psi^2(a^k) = a^{-k}$. Therefore, we can assume (up to equivalence of $\mathcal{M}$) that

$$\psi(a^{-1}) = b \text{ and } \psi(b) = a.$$

Case 1. $p^{D_n/T} = (Tb, Ta, Ta^2 b, Ta^3, \ldots, Ta^{m-2}b, Ta^{m-1})$.

Clearly, $p(a) \in \{a^2 b, a^{m+2}b\}$ and $p^2(a) \in \{a^3, a^{m+3}\}$. Below we go through all possibilities.

Subcase 1.1. $p(a) = a^2 b$ and $p(a^2 b) = a^3$. Then $\psi(ab) = \psi(a)\psi^{\pi(a)}(b) = a^2 b\psi^3(b) = a^2 ba^3 = a^{-1}b$. Thus $a^{-1}b = \psi(ab) = \psi(ba^{-1}) = \psi(b)\psi^{\pi(b)}(a^{-1}) = a\psi^{-1}(a^{-1})$, which implies that $\psi^{-1}(a^{-1}) = a^{-2}b$, and so $\psi(a^{-2}b) = a^{-1}$. Now we have $a^{-1} = \psi(a^{-2}b) = \psi(ba^2) = \psi(b)\psi^{\pi(b)}(a^2) = a\psi^{-1}(a^2)$, and $a^3 = \psi(a^2 b) = \psi(ba^{-2}) = \psi(b)\psi^{\pi(b)}(a^{-2}) = $

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$a\psi^{-1}(a^{-2})$. Consequently, ψ switches a^2 and a^{n-2}. By this one can prove that $\psi(a^{2i}) = a^{-2i}$ for every $i \in \{1, \ldots, m\}$. Since Π contains only odd numbers, we find all the remaining values as:

$$
\begin{aligned}
\psi(a^{2i+1}b) &= \psi(aba^{-2i}) = \psi(ab)\psi^{\pi(ab)}(a^{-2i}) = a^{-1}ba^{2i} = a^{n-2i-1}b \\
\psi(a^{2i+1}) &= \psi(a)\psi^{\pi(a)}(a^{2i}) = a^2ba^{-2i} = a^{2i+2}b \\
\psi(a^{2i}b) &= \psi(ba^{-2i}) = \psi(b)\psi^{\pi(b)}\psi(a^{-2i}) = aa^{2i} = a^{2i+1}
\end{aligned}
$$

Thus ψ is the skew-morphism given in Lemma 4.4(1), and case (4) follows.

Subcase 1.2. $p(a) = a^2b$ and $p^2(a) = a^{m+3}$. Then $\psi(ab) = \psi(a)\psi^{\pi(a)}(b) = a^2b\psi^3(b) = a^2ba^{m+3} = a^{m-1}b$. Thus $a^{m-1}b = \psi(ab) = \psi(ba^{-1}) = \psi(b)\psi^{\pi(b)}(a^{-1}) = a\psi^{-1}(a^{-1})$, which implies that $\psi^{-1}(a^{-1}) = a^{m-2}b$, and so $\psi(a^{-2}b) = a^{m-1}$. Then we may write $a^{m-1} = \psi(a^{-2}b) = \psi(ba^2) = \psi(b)\psi^{\pi(b)}(a^2) = a\psi^{-1}(a^2)$ and $a^{m+3} = \psi(a^2b) = \psi(ba^{-2}) = \psi(b)\psi^{\pi(b)}(a^{-2}) = a\psi^{-1}(a^{-2})$. Consequently, ψ switches a^2 and a^{m-2}, and also a^{-2} and a^{m+2}. Now $a^2b = \psi(a) = \psi(a^{-1}a^2) = \psi(a^{-1})\psi^{\pi(a^{-1})}(a^2) = ba^{m-2}$, a contradiction.

Subcase 1.3. $p(a) = a^{m+2}b$ and $p^2(a) = a^3$. Then $\psi(ab) = \psi(a)\psi^{\pi(a)}(b) = a^{m+2}b\psi^3(b) = a^{m+2}ba^3 = a^{m-1}b$. Thus $a^{m-1}b = \psi(ab) = \psi(ba^{-1}) = \psi(b)\psi^{\pi(b)}(a^{-1}) = a\psi^{-1}(a^{-1})$, which means $\psi^{-1}(a^{-1}) = a^{m-2}b$, and so $\psi(a^{m-2}b) = a^{-1}$. Now $\psi(a^2) = \psi(a^{-1}a^3) = \psi(a^{-1})\psi^{\pi(a^{-1})}(a^3) = ba^{m+2}b = a^{m-2}$, which implies $\psi(a^{m+2}) = a^{-2}$. Also, we have $\psi(a^{-2}) = \psi(a^{-1}a^{-1}) = \psi(a^{-1})\psi^{\pi(a^{-1})}(a^{-1}) = ba^{m-2}b = a^{m+2}$, and hence $\psi(a^{m-2}) = a^2$. Furthermore, $\psi(a^4) = \psi(a^2)\psi^{\pi(a^2)}(a^2) = a^{m-2}a^{m-2} = a^{-4}$; and similarly, $\psi(a^{-4}) = a^4$. By these one can prove that $\psi(a^{4i+2}) = a^{-4i-2+m}$ and $\psi(a^{4i}) = a^{-4i}$ for every $i \in \{1, \ldots, m/2 - 1\}$. Now for ψ to be well defined, m should be a multiple of 4, and hence $8 \mid n$. Using these values of ψ and the fact that Π contains only odd numbers, we find all the remaining values as:

$$
\begin{aligned}
\psi(a^{2i+1}b) &= \psi(aba^{-2i}) = \psi(ab)\psi^{\pi(ab)}(a^{-2i}) = a^{m-1}ba^{im+2i} = a^{(i+1)m-2i-1}b \\
\psi(a^{2i+1}) &= \psi(a)\psi^{\pi(a)}(a^{2i}) = a^{m+2}ba^{im-2i} = a^{(i+1)m+2i+2}b \\
\psi(a^{2i}b) &= \psi(ba^{-2i}) = \psi(b)\psi^{\pi(b)}\psi(a^{-2i}) = aa^{im+2i} = a^{im+2i+1}
\end{aligned}
$$

Thus ψ is the skew-morphism given in Lemma 4.5(1), and case (5) follows.

Subcase 1.4. $p(a) = a^{m+2}b$ and $p^2(a) = a^{m+3}$. Then $\psi(ab) = \psi(a)\psi^{\pi(a)}(b) = a^{m+2}b\psi^3(b) = a^{m+2}ba^{m+3} = a^{-1}b$. Thus $a^{-1}b = \psi(ab) = \psi(ba^{-1}) = \psi(b)\psi^{\pi(b)}(a^{-1}) = a\psi^{-1}(a^{-1})$, which means $\psi^{-1}(a^{-1}) = a^{-2}b$, and so $\psi(a^{-2}b) = a^{-1}$. Now we have $a^{-1} = \psi(a^{-2}b) = \psi(ba^2) = \psi(b)\psi^{\pi(b)}(a^2) = a\psi^{-1}(a^2)$ and $a^{m+3} = \psi(a^{m+2}b) = \psi(ba^{m-2}) = \psi(b)\psi^{\pi(b)}(a^{m-2}) = a\psi^{-1}(a^{m-2})$. Consequently, ψ switches a^2 and a^{-2}, and also a^{m+2} and a^{m-2}. Now, $a^{m+2}b = \psi(a) = \psi(a^{-1}a^2) = \psi(a^{-1})\psi^{\pi(a^{-1})}(a^2) = ba^{-2}$, a contradiction.

Case 2. $p^{D_n/T} = (Tb, Ta, Ta^{2+m/2}b, \ldots, Ta^{m-1})$.

Now $\psi^{-1}(a^{-1}) \in \{a^{\frac{m}{2}-2}b, a^{\frac{3m}{2}-2}b\}$ and $\pi(\psi^{-1}(a^{-1})) = 3$. By Lemma 5.2(3),

$$
\pi(\psi^{-1}(a^{-1})a^{-1}) \equiv \pi(\psi^{-1}(a^{-1})) + \pi(a^{-1}) - 1 \equiv 1 (\bmod \, n),
$$

which implies that $\psi^{-1}(a^{-1})a^{-1} \in \mathrm{Ker}(\pi)$. Notice that $\psi^{-1}(a^{-1})a^{-1} = a^{\frac{m}{2}-1}b$ or $a^{\frac{3m}{2}-1}b$. Since $\pi(a^{-1}b) = \pi(ba) \equiv \pi(b) + \pi(a) - 1 \equiv 1(\mathrm{mod}\ n)$, $a^{-1}b$ also belongs to $\mathrm{Ker}(\pi)$. These imply that $a^{\frac{m}{2}} \in \mathrm{Ker}(\pi)$, and hence $|\mathrm{Ker}(\pi)| \geq 8$, a contradiction. $\qquad\square$

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